# MONOMORPHISM CATEGORIES, COTILTING THEORY, AND GORENSTEIN-PROJECTIVE MODULES

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*Dedicated to Claus Michael Ringel on the occasion of his $65^{th}$ birthday*

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Abstract. The monomorphism category $\mathcal{S}_n(\mathcal{X})$ is introduced, where $\mathcal{X}$ is a full subcategory of the module category $A$-mod of Artin algebra $A$. The key result is a reciprocity of the monomorphism operator $\mathcal{S}_n$ and the left perpendicular operator ${}^\perp$: for a cotilting $A$-module $T$, there is a canonical construction of a cotilting $T_n(A)$-module $\mathbf{m}(T)$, such that $\mathcal{S}_n({}^\perp T) = {}^\perp\mathbf{m}(T)$.

As applications, $\mathcal{S}_n(\mathcal{X})$ is a resolving contravariantly finite subcategory in $T_n(A)$-mod with $\widehat{\mathcal{S}_n(\mathcal{X})} = T_n(A)$-mod if and only if $\mathcal{X}$ is a resolving contravariantly finite subcategory in $A$-mod with $\widehat{\mathcal{X}} = A$-mod. For a Gorenstein algebra $A$, the category $T_n(A)$-$\mathcal{G}proj$ of Gorenstein-projective $T_n(A)$-modules can be explicitly determined as $\mathcal{S}_n({}^\perp A)$. Also, self-injective algebras $A$ can be characterized by the property $T_n(A)$-$\mathcal{G}proj = \mathcal{S}_n(A)$. Using $\mathcal{S}_n(A) = {}^\perp\mathbf{m}(D(A_A))$, a characterization of $\mathcal{S}_n(A)$ of finite type is obtained.

*Key words and phrases. Monomorphism category, cotilting modules, Gorenstein-projective modules*

## Introduction

Throughout $A$ is an Artin algebra, and $n \geq 2$ an integer. Let $A$-mod be the category of finitely generated left $A$-modules, and $\mathcal{X}$ a full subcategory of $A$-mod. Denote by $\mathrm{Mor}_n(A)$ the morphism category, which is equivalent to $T_n(A)$-mod, where $T_n(A)$ is the upper triangular matrix algebra of $A$. Let $\mathcal{S}_n(A)$ denote the full subcategory of $\mathrm{Mor}_n(A)$ given by $\mathcal{S}_n(A) = \{X_{(\phi_i)} \in \mathrm{Mor}_n(A) \mid \text{all } \phi_i \text{ are monomorphisms}\}$.

G. Birkhoff [Bir] initiated to classify the indecomposable objects of $\mathcal{S}_2(\mathbb{Z}/\langle p^t\rangle)$. In [RW] the indecomposable objects of $\mathcal{S}_2(\mathbb{Z}/\langle p^t\rangle)$ with $t \leq 5$ were determined. In [Ar] $\mathcal{S}_n(R)$ was denoted by $\mathcal{C}(n, R)$, where $R$ is a commutative uniserial artinian ring; and the complete lists of $\mathcal{C}(n, R)$ of finite type, and of the representation types of $\mathcal{C}(n, k[x]/\langle x^t\rangle)$, have been given by D. Simson [S] (see also [SW]). C. M. Ringel and M. Schmidmeier ([RS1] - [RS3]) have intensively studied the monomorphism category $\mathcal{S}_2(A)$. According to [RS2], $\mathcal{S}_n(A)$ is a functorially finite subcategory in $T_n(A)$-mod; and hence $\mathcal{S}_n(A)$ has Auslander-Reiten sequences. For more recent work related to the monomorphism categories we refer to [C], [IKM] and [KLM].

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*2010 Mathematical Subject Classification.* 16G10, 16E65, 16G50.
Supported by the NSF of China (10725104), and STCSM (09XD1402500).
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On the other hand, M. Auslander and I. Reiten [AR] have established a relation between resolving contravariantly finite subcategories and cotilting theory, by asserting that $\mathcal{X}$ is resolving and contravariantly finite with $\widehat{\mathcal{X}} = A$-mod if and only if $\mathcal{X} = {}^{\perp}T$ for some cotilting $A$-module $T$ ([AR], Theorem 5.5(a)), where ${}^{\perp}T$ is the left perpendicular category of $T$.

Define $\mathcal{S}_n(\mathcal{X})$ to be the full subcategory of $\mathrm{Mor}_n(A)$ of all the objects $X_{(\phi_i)}$, where all $X_i \in \mathcal{X}$, all $\phi_i$ are monomorphisms, and all $\mathrm{Coker}\,\phi_i \in \mathcal{X}$. A main problem we concern is: when is $\mathcal{S}_n(\mathcal{X})$ contravariantly finite in $T_n(A)$-mod? This leads to the following reciprocity of the monomorphism operator $\mathcal{S}_n$ and the left perpendicular operator ${}^{\perp}$: given a cotilting $A$-module $T$, then $\mathbf{m}(T) = \begin{pmatrix} T \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \oplus \begin{pmatrix} T \\ T \\ 0 \\ \vdots \\ 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} T \\ T \\ T \\ \vdots \\ T \end{pmatrix}$ is a cotilting $T_n(A)$-module, such that $\mathcal{S}_n({}^{\perp}T) = {}^{\perp}\mathbf{m}(T)$. See Theorem 3.1. The proof needs the contravariantly finiteness of $\mathcal{S}_n(A)$ in $T_n(A)$-mod, and the six adjoint pairs between $A$-mod and $T_n(A)$-mod.

We illustrate this reciprocity with several applications. First, we have a solution to the main problem: $\mathcal{S}_n(\mathcal{X})$ is a resolving contravariantly finite subcategory in $T_n(A)$-mod with $\widehat{\mathcal{S}_n(\mathcal{X})} = T_n(A)$-mod if and only if $\mathcal{X}$ is a resolving contravariantly finite subcategory in $A$-mod with $\widehat{\mathcal{X}} = A$-mod (Theorem 3.9).

As another application, taking $T = {}_AA$ for Gorenstein algebra $A$, the category $T_n(A)$-$\mathcal{G}proj$ of Gorenstein-projective $T_n(A)$-modules, can be determined explicitly as $\mathcal{S}_n({}^{\perp}A)$ (Corollary 4.1). By D. Happel's triangle-equivalence $D^b(A)/K^b(\mathcal{P}(A)) \cong \underline{{}^{\perp}A}$ for Gorenstein algebra $A$, one has $D^b(T_n(A))/K^b(\mathcal{P}(T_n(A))) = \underline{\mathcal{S}_n({}^{\perp}A)}$ (Corollary 4.3). Also, self-injective algebras $A$ can be characterized by the property $\mathcal{S}_n(A) = T_n(A)$-$\mathcal{G}proj$ (Theorem 4.4).

The representation type of $\mathcal{S}_n(A)$ is quite different from the ones of $A$ and of $T_n(A)$. For example, $k[x]/\langle x^t\rangle$ is of finite type, $T_2(k[x]/\langle x^t\rangle)$ is of finite type if and only if $t \le 3$, but $\mathcal{S}_2(k[x]/\langle x^t\rangle)$ is of finite type if and only if $t \le 5$, where $k$ is an algebraically closed field. If $t > 6$ then $\mathcal{S}_2(k[x]/\langle x^t\rangle)$ is of "wild" type, while $\mathcal{S}_2(k[x]/\langle x^6\rangle)$ is of "tame" type ([S], Theorems 5.2 and 5.5). A complete classification of indecomposable objects of $\mathcal{S}_2(k[x]/\langle x^6\rangle)$ is exhibited in [RS3]. Inspired by Auslander's classical result: $A$ is of finite type if and only if there is an $A$-generator-cogenerator $M$ such that $\mathrm{gl.dim}\,\mathrm{End}_A(M) \le 2$ ([Au], Chapter III), by using $\mathcal{S}_n(A) = {}^{\perp}\mathbf{m}(D(A_A))$, we prove that $\mathcal{S}_n(A)$ is of finite type if and only if there is a bi-generator $M$ of $\mathcal{S}_n(A)$ such that $\mathrm{gl.dim}\,\mathrm{End}_{T_n(A)}(M) \le 2$ (Theorem 5.1). As a corollary, for a self-injective algebra $A$, $T_n(A)$ is CM-finite if and only if there is a $T_n(A)$-generator $M$ which is Gorenstein-projective, such that $\mathrm{gl.dim}\,\mathrm{End}_{T_n(A)}(M) \le 2$ (Corollary 5.2).

## 1. Monomorphism categories

We will define the monomorphism category $\mathcal{S}_n(\mathcal{X})$ and give its basic properties needed later.

1.1. An object of the morphism category $\mathrm{Mor}_n(A)$ is $X_{(\phi_i)} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}_{(\phi_i)}$, where $\phi_i : X_{i+1} \to X_i$

is an $A$-map, $1 \le i \le n-1$; and a morphism $X_{(\phi_i)} \to Y_{(\theta_i)}$ is $f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$, where $f_i : X_i \to Y_i$ is an $A$-map, $1 \le i \le n$, such that every square in the following diagram commutes

$$\begin{tikzcd} X_n \arrow[r, "\phi_{n-1}"] \arrow[d, "f_n"'] & X_{n-1} \arrow[r, "\phi_{n-2}"] \arrow[d, "f_{n-1}"'] & \cdots \arrow[r, "\phi_1"] & X_1 \arrow[d, "f_1"'] \\ Y_n \arrow[r, "\theta_{n-1}"] & Y_{n-1} \arrow[r, "\theta_{n-2}"] & \cdots \arrow[r, "\theta_1"] & Y_1. \end{tikzcd} \tag{1.1}$$

Note that $\mathrm{Mor}_n(A)$ is again an abelian category, and that a sequence $Z_{(\psi_i)} \xrightarrow{f} Y_{(\theta_i)} \xrightarrow{g} X_{(\phi_i)}$ in $\mathrm{Mor}_n(A)$ is exact if and only if $Z_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} X_i$ is exact in $A$-mod for each $1 \le i \le n$.

1.2. We define $\mathcal{S}_n(\mathcal{X})$ to be the full subcategory of $\mathrm{Mor}_n(A)$ consisting of all the objects $X_{(\phi_i)}$, where $X_i \in \mathcal{X}$ for $1 \le i \le n$, $\phi_i : X_{i+1} \hookrightarrow X_i$ is a monomorphism and $\mathrm{Coker}\,\phi_i \in \mathcal{X}$ for $1 \le i \le n-1$. In particular, we have $\mathcal{S}_n(A\text{-mod}) = \{X_{(\phi_i)} \in \mathrm{Mor}_n(A) \mid \phi_i \text{ is monic}, 1 \le i \le n-1\}$, which will be denoted by $\mathcal{S}_n(A)$. Dually, $\mathcal{F}_n(\mathcal{X})$ is the full subcategory of $\mathrm{Mor}_n(A)$ consisting of all the objects $X_{(\phi_i)}$, where $X_i \in \mathcal{X}$, $1 \le i \le n$, $\phi_i :\ X_{i+1} \twoheadrightarrow X_i$ is an epimorphism and $\mathrm{Ker}\,\phi_i \in \mathcal{X}$ for $1 \le i \le n-1$. We call $\mathcal{S}_n(\mathcal{X})$ and $\mathcal{F}_n(\mathcal{X})$ *the monomorphism category* and *the epimorphism category of* $\mathcal{X}$, respectively.

**Lemma 1.1.** *Let $A$ be an Artin algebra and $\mathcal{X}$ a full subcategory of $A$-mod.*

*(i) Let $0 \to Z_{(\psi_i)} \to Y_{(\theta_i)} \to X_{(\phi_i)} \to 0$ be an exact sequence in $\mathrm{Mor}_n(A)$. Then the following induced sequences are exact for each $1 \le i \le n-1$*

$$\begin{aligned} 0 \longrightarrow \mathrm{Ker}(\psi_1 \cdots \psi_i) \longrightarrow \mathrm{Ker}(\theta_1 \cdots \theta_i) \longrightarrow \mathrm{Ker}(\phi_1 \cdots \phi_i) \longrightarrow \\ \longrightarrow \mathrm{Coker}(\psi_1 \cdots \psi_i) \longrightarrow \mathrm{Coker}(\theta_1 \cdots \theta_i) \longrightarrow \mathrm{Coker}(\phi_1 \cdots \phi_i) \longrightarrow 0, \end{aligned}$$

*and*

$$0 \longrightarrow \mathrm{Ker}\,\psi_i \longrightarrow \mathrm{Ker}\,\theta_i \longrightarrow \mathrm{Ker}\,\phi_i \longrightarrow \mathrm{Coker}\,\psi_i \longrightarrow \mathrm{Coker}\,\theta_i \longrightarrow \mathrm{Coker}\,\phi_i \longrightarrow 0.$$

*(ii) $\mathcal{S}_n(\mathcal{X})$ is closed under extensions (resp., kernels of epimorphisms, direct summands) if and only if $\mathcal{X}$ is closed under extensions (resp., kernels of epimorphisms, direct summands).*

*(iii) $\mathcal{S}_n(A)$ is closed under subobjects.*

*(iv) If $\mathcal{X}$ is closed under extensions, then there is an equivalence of categories $\mathcal{S}_n(\mathcal{X}) \cong \mathcal{F}_n(\mathcal{X})$ given by*

$$\mathcal{S}_n(\mathcal{X}) \ni \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{pmatrix}_{(\phi_i)} \mapsto \begin{pmatrix} \mathrm{Coker}\,\phi_1 \\ \mathrm{Coker}(\phi_1\phi_2) \\ \vdots \\ \mathrm{Coker}(\phi_1 \cdots \phi_{n-1}) \\ X_1 \end{pmatrix}_{(\phi'_i)}$$

*where $\phi'_i : \mathrm{Coker}(\phi_1 \cdots \phi_{i+1}) \twoheadrightarrow \mathrm{Coker}(\phi_1 \cdots \phi_i)$, $1 \le i \le n-2$, and $\phi'_{n-1} : X_1 \twoheadrightarrow \mathrm{Coker}(\phi_1 \cdots \phi_{n-1})$, are the canonical epimorphisms, with a quasi-inverse*

$$\mathcal{F}_n(\mathcal{X}) \ni \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{pmatrix}_{(\phi_i)} \mapsto \begin{pmatrix} X_n \\ \mathrm{Ker}(\phi_1 \cdots \phi_{n-1}) \\ \vdots \\ \mathrm{Ker}(\phi_{n-2}\phi_{n-1}) \\ \mathrm{Ker}\,\phi_{n-1} \end{pmatrix}_{(\phi''_i)},$$

*where* $\phi''_i : \mathrm{Ker}(\phi_i \cdots \phi_{n-1}) \hookrightarrow \mathrm{Ker}(\phi_{i-1} \cdots \phi_{n-1}),\ 2 \le i \le n-1,$ *and* $\phi''_1 : \mathrm{Ker}(\phi_1 \cdots \phi_{n-1}) \hookrightarrow X_n$, *are the canonical monomorphisms.*

**Proof.** Applying Snake Lemma to the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & Z_{i+1} \arrow[r] \arrow[d, "\psi_1 \cdots \psi_i"'] & Y_{i+1} \arrow[r] \arrow[d, "\theta_1 \cdots \theta_i"] & X_{i+1} \arrow[r] \arrow[d, "\phi_1 \cdots \phi_i"] & 0 \\
0 \arrow[r] & Z_1 \arrow[r] & Y_1 \arrow[r] & X_1 \arrow[r] & 0
\end{tikzcd}$$

we get the first exact sequence in $(i)$; and the second one can be similarly obtained. $(ii)$ follows from $(i)$; $(iii)$ can seen from (1.1), and $(iv)$ is clear. ■

1.3. Let $X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}_{(\phi_i)} \in \mathrm{Mor}_n(A)$. We call $X_i$ the $i$-th branch of $X$, and $\phi_i$ the $i$-th morphism of $X$. For each $1 \le i \le n$, we define a functor $\mathbf{m}_i : A\text{-mod} \to \mathcal{S}_n(A)$ as follows. For $M \in A$-mod, the $j$-th branch of $\mathbf{m}_i(M)$ is $M$ if $j \le i$, and 0 if $j > i$; and the $j$-th morphism of $\mathbf{m}_i(M)$ is $\mathrm{id}_M$ if $j < i$, and 0 if $j \ge i$. For each $A$-map $f : M \to N$, we define

$$\mathbf{m}_i(f) = \begin{pmatrix} f \\ \vdots \\ f \\ 0 \\ \vdots \\ 0 \end{pmatrix} : \ \mathbf{m}_i(M) = \begin{pmatrix} M \\ \vdots \\ M \\ 0 \\ \vdots \\ 0 \end{pmatrix} \longrightarrow \mathbf{m}_i(N) = \begin{pmatrix} N \\ \vdots \\ N \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Note that the restriction of $\mathbf{m}_i$ to $\mathcal{X}$ gives a functor $\mathcal{X} \to \mathcal{S}_n(\mathcal{X})$. A functor $\mathbf{p}_i : A\text{-mod} \to \mathcal{F}_n(A)$ is dually defined, $1 \le i \le n$. The $j$-th branch of $\mathbf{p}_i(M)$ is $M$ if $j \ge n-i+1$, and 0 if $j < n-i+1$; and the $j$-th morphism of $\mathbf{p}_i(M)$ is $\mathrm{id}_M$ if $j \ge n-i+1$, and 0 if $j < n-i+1$. Also we define

$$\mathbf{p}_i(f) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f \\ \vdots \\ f \end{pmatrix} : \ \mathbf{p}_i(M) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ M \\ \vdots \\ M \end{pmatrix} \longrightarrow \mathbf{p}_i(N) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ N \\ \vdots \\ N \end{pmatrix}.$$

The restriction of $\mathbf{p}_i$ to $\mathcal{X}$ gives a functor $\mathcal{X} \to \mathcal{F}_n(\mathcal{X})$. We have $\mathbf{m}_n(M) = \mathbf{p}_n(M)$, $\forall\, M \in A$-mod.

The following facts imply that in fact there are six adjoint pairs between $A$-mod and $\mathrm{Mor}_n(A)$.

**Lemma 1.2.** *Let $A$ be an Artin algebra. Then for each object $X = X_{(\phi_i)} \in \mathrm{Mor}_n(A)$ and each $A$-module $M$, we have isomorphisms of abelian groups, which are natural in both positions*

$$\mathrm{Hom}_{\mathrm{Mor}_n(A)}(\mathbf{m}_i(M), X) \cong \mathrm{Hom}_A(M, X_i), \quad 1 \le i \le n, \tag{1.2}$$

$$\mathrm{Hom}_{\mathrm{Mor}_n(A)}(X, \mathbf{m}_i(M)) \cong \mathrm{Hom}_A(\mathrm{Coker}(\phi_1 \cdots \phi_i), M), \quad 1 \le i \le n-1, \tag{1.3}$$

$$\mathrm{Hom}_{\mathrm{Mor}_n(A)}(X, \mathbf{m}_n(M)) \cong \mathrm{Hom}_A(X_1, M), \tag{1.4}$$

$$\mathrm{Hom}_{\mathrm{Mor}_n(A)}(X, \mathbf{p}_i(M)) \cong \mathrm{Hom}_A(X_{n-i+1}, M), \quad 1 \le i \le n, \tag{1.5}$$

$$\mathrm{Hom}_{\mathrm{Mor}_n(A)}(\mathbf{p}_i(M), X) \cong \mathrm{Hom}_A(M, \mathrm{Ker}(\phi_{n-i} \cdots \phi_{n-1})), \quad 1 \le i \le n-1, \tag{1.6}$$

$$\mathrm{Hom}_{\mathrm{Mor}_n(A)}(\mathbf{p}_n(M), X) \cong \mathrm{Hom}_A(M, X_n). \tag{1.7}$$

**Proof.** We justify (1.3). Let $\pi_i : X_1 \twoheadrightarrow \operatorname{Coker}(\phi_1 \cdots \phi_i)$ be the canonical epimorphism, $1 \le i \le n-1$. Consider the homomorphism of abelian groups $\operatorname{Hom}_A(\operatorname{Coker}(\phi_1 \cdots \phi_i), M) \to \operatorname{Hom}_{\operatorname{Mor}_n(A)}(X, \mathbf{m}_i(M))$ given by

$$g \mapsto \begin{pmatrix} g\pi_i \\ g\pi_i\phi_1 \\ \vdots \\ g\pi_i\phi_1\cdots\phi_{i-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} : \ X \longrightarrow \begin{pmatrix} M \\ M \\ \vdots \\ M \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \forall\, g \in \operatorname{Hom}_A(\operatorname{Coker}(\phi_1 \cdots \phi_i), M).$$

By (1.1) we infer that it is surjective, and it is injective since $\pi$ is epic. It is clear that the isomorphisms are natural in both positions. ■

1.4. Let $\mathcal{P}(A)$ (resp. $\mathcal{I}(A)$) be the full subcategory of $A$-mod of projective (resp. injective) $A$-modules, and $\operatorname{Ind}\mathcal{P}(A)$ (resp. $\operatorname{Ind}\mathcal{I}(A)$) be the set of pairwise non-isomorphic indecomposable projective (resp. injective) $A$-modules.

**Lemma 1.3.** *Let $A$ be an Artin algebra. Then*

(*i*) *There is an equivalence of categories* $\operatorname{Mor}_n(A) \cong T_n(A)$-mod, *which preserves the exact structures, where $T_n(A)$ is the $n \times n$ upper triangular matrix algebra* $\begin{pmatrix} A & A & \cdots & A \\ 0 & A & \cdots & A \\ & & \ddots & \\ 0 & 0 & \cdots & A \end{pmatrix}$.

(*ii*) *Under this equivalence, we have*

$$\operatorname{Ind}\mathcal{P}(T_n(A)) = \{\mathbf{m}_1(P), \ \cdots, \ \mathbf{m}_n(P) \mid P \in \operatorname{Ind}\mathcal{P}(A)\} \subseteq \mathcal{S}_n(A), \tag{1.8}$$

$$\operatorname{Ind}\mathcal{I}(T_n(A)) = \{\mathbf{p}_1(I), \ \cdots, \ \mathbf{p}_n(I) \mid I \in \operatorname{Ind}\mathcal{I}(A)\} \subseteq \mathcal{F}_n(A). \tag{1.9}$$

**Proof.** (*i*) This is well-known, at least for $n = 2$ (see [ARS], p.71). For convenience we include a short justification. For $1 \le i \le j \le n$, let $e_{ij} \in T_n(A)$ be the matrix with 1 in the $(i, j)$-entry, and 0 elsewhere. For a $T_n(A)$-module $M$ we have $M = e_{11}M \oplus \cdots \oplus e_{nn}M$ as $A$-modules, and for an $A$-map $f : M \to N$, the restriction $f_i$ of $f$ to $e_{ii}M$ gives an $A$-map $f_i : e_{ii}M \to e_{ii}N$. Consider a functor $F : T_n(A)\text{-mod} \to \operatorname{Mor}_n(A)$ defined by $F(M) = \begin{pmatrix} e_{11}M \\ \vdots \\ e_{nn}M \end{pmatrix}_{(\phi_{M,i})}$, where $\phi_{M,i} : e_{i+1i+1}M \to e_{ii}M$ is the $A$-map given by $\phi_{M,i}(e_{i+1i+1}x) = e_{ii}e_{ii+1}e_{i+1i+1}x \in e_{ii}M, \ 1 \le i \le n-1$, and

$$F(f) = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} : \begin{pmatrix} e_{11}M \\ \vdots \\ e_{nn}M \end{pmatrix}_{(\phi_{M,i})} \longrightarrow \begin{pmatrix} e_{11}N \\ \vdots \\ e_{nn}N \end{pmatrix}_{(\phi_{N,i})}.$$

Then $F$ is fully faithful. For each object $\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}_{(\phi_i)} \in \operatorname{Mor}_n(A)$, put $X = \bigoplus\limits_{1 \le i \le n} X_i$, and write an element of $X$ as $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ with $x_i \in X_i$. With a $T_n(A)$-action on $X$ defined by

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ & & \ddots & \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \sum\limits_{j>1} a_{1j}\phi_1\cdots\phi_{j-1}(x_j) \\ \vdots \\ a_{ii}x_i + \sum\limits_{j>i} a_{ij}\phi_i\cdots\phi_{j-1}(x_j) \\ \vdots \\ a_{nn}x_n \end{pmatrix},$$

$X$ is a $T_n(A)$-module such that $F(X) = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}_{(\phi_i)}$, i.e., $F$ is dense.

$(ii)$ Since $\mathrm{End}_{T_n(A)}(\mathbf{m}_i(M)) \cong \mathrm{End}_A(M) \cong \mathrm{End}_{T_n(A)}(\mathbf{p}_i(M))$, $\forall\ M \in A$-mod, it follows that if $M$ is indecomposable then $\mathbf{m}_i(M)$ and $\mathbf{p}_i(M)$ are indecomposable. Since $\mathbf{m}_i$ and $\mathbf{p}_i$ are additive functors, (1.8) follows from the decomposition $T_n(A) = \bigoplus\limits_{1\le i\le n} \mathbf{m}_i(A)$ as left $T_n(A)$-modules. By (1.5) we see that $\mathbf{p}_i(I)$, $1 \le i \le n$, are indecomposable injective $T_n(A)$-modules, where $I \in \mathrm{Ind}\mathcal{I}(A)$, and then we infer (1.9), by comparing the number of pairwise non-isomorphic indecomposable injective $T_n(A)$-modules. ■

From now on we identify $T_n(A)$-mod with $\mathrm{Mor}_n(A)$.

1.5. A full subcategory $\mathcal{X}$ of $A$-mod is *resolving* if $\mathcal{X}$ contains all projective $A$-modules, $\mathcal{X}$ is closed under extensions, kernels of epimorphisms, and direct summands. By Lemmas 1.3$(ii)$ and 1.1$(ii)$, and using functor $\mathbf{m}_1 : \mathcal{X} \to \mathcal{S}_n(\mathcal{X})$ we get

**Corollary 1.4.** *Let $A$ be an Artin algebra and $\mathcal{X}$ a full subcategory of $A$-mod. Then $\mathcal{S}_n(\mathcal{X})$ is a resolving subcategory of $\mathrm{Mor}_n(A)$ if and only if $\mathcal{X}$ is a resolving subcategory of $A$-mod.*

## 2. Functorially finiteness of $\mathcal{S}_n(A)$ in $\mathrm{Mor}_n(A)$

The idea of the following result comes from [RS2] for $\mathcal{S}_2(A)$.

**Theorem 2.1.** *(Ringel - Schmidmeier) Let $A$ be an Artin algebra. Then $\mathcal{S}_n(A)$ is a functorially finite subcategory in $\mathrm{Mor}_n(A)$ and has Auslander-Reiten sequences.*

2.1. Let $X_{(\phi_i)} \in \mathrm{Mor}_n(A)$. Fix an injective envelope $e'_i : \mathrm{Ker}\,\phi_i \hookrightarrow \mathrm{IKer}\,\phi_i$. Define object $\mathrm{rMon}(X) \in \mathcal{S}_n(A)$ as follows. We have an $A$-map $e_i : X_{i+1} \to \mathrm{IKer}\,\phi_i$ such that the following diagram commutes for each $1 \le i \le n-1$

$$\begin{tikzcd} \mathrm{Ker}\,\phi_i \arrow[r, hook] \arrow[d, hook, "e'_i"'] & X_{i+1} \arrow[dl, "e_i"] \\ \mathrm{IKer}\,\phi_i & \end{tikzcd} \qquad . \tag{2.1}$$

Of course $e_1, \cdots, e_{n-1}$ are not unique. However we choose and fix them, and then define $\theta_i : X_{i+1} \oplus \mathrm{IKer}\,\phi_{i+1} \oplus \cdots \oplus \mathrm{IKer}\,\phi_{n-1} \longrightarrow X_i \oplus \mathrm{IKer}\,\phi_i \oplus \mathrm{IKer}\,\phi_{i+1} \oplus \cdots \oplus \mathrm{IKer}\,\phi_{n-1}$ to be

$$\theta_i = \begin{pmatrix} \phi_i & 0 & 0 & \cdots & 0 \\ e_i & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{(n-i+1)\times(n-i)}, \tag{2.2}$$

and define

$$\mathrm{rMon}(X) = \begin{pmatrix} X_1 \oplus \mathrm{IKer}\,\phi_1 \oplus \cdots \oplus \mathrm{IKer}\,\phi_{n-1} \\ X_2 \oplus \mathrm{IKer}\,\phi_2 \oplus \cdots \oplus \mathrm{IKer}\,\phi_{n-1} \\ \vdots \\ X_{n-1} \oplus \mathrm{IKer}\,\phi_{n-1} \\ X_n \end{pmatrix}_{(\theta_i)} . \tag{2.3}$$

By construction all $\theta_i$'s are monomorphisms, and hence $\mathrm{rMon}(X) \in \mathcal{S}_n(A)$.

**Remark 2.2.** *By definition* $\mathrm{rMon}(X)$ *seems to be of dependent on the choices of* $e_1, \cdots, e_{n-1}$. *However, for an arbitrary choice of* $e_1, \cdots, e_{n-1}$, $\mathrm{rMon}(X)$ *will be proved to be a right minimal approximation of* $X$ *in* $\mathcal{S}_n(A)$. *Thus, by the uniqueness of a right minimal approximation,* $\mathrm{rMon}(X)$ *is in fact independent of the choices of* $e_1, \cdots, e_{n-1}$, *up to isomorphism in* $\mathrm{Mor}_n(A)$.

2.2. Denote by $\widehat{\mathcal{X}}$ the full subcategory of $A$-mod given by ([AR])

$$\widehat{\mathcal{X}} = \{X \in A\text{-mod} \mid \exists \text{ an exact sequence } 0 \to X_m \to \cdots \to X_0 \to X \to 0 \text{ with } X_i \in \mathcal{X},\ 0 \le i \le m\}.$$

A morphism $f : X \to M$ is *right minimal*, if every endomorphism $g$ of $X$ with $fg = f$ is an isomorphism. *A right approximation of* $M$ *in* $\mathcal{X}$ is a morphism $f : X \to M$ with $X \in \mathcal{X}$, such that the induced homomorphism $\mathrm{Hom}_A(X', X) \to \mathrm{Hom}_A(X', M)$ is surjective for each $X' \in \mathcal{X}$. A right approximation $f : X \to M$ is *a right minimal approximation* if $f$ is right minimal. If every objet $M$ admits a right minimal approximation in $\mathcal{X}$, then $\mathcal{X}$ is called *a contravariantly finite subcategory in* $A$-mod. Dually we have *a covariantly finite subcategory in* $A$-mod. If $\mathcal{X}$ is both contravariantly and covariantly finite in $A$-mod, then $\mathcal{X}$ is *a functorially finite subcategory in* $A$-mod.

**Lemma 2.3.** *Let* $A$ *be an Artin algebra. Then* $\mathcal{S}_n(A)$ *is a contravariantly finite subcategory in* $\mathrm{Mor}_n(A)$ *with* $\widehat{\mathcal{S}_n(A)} = \mathrm{Mor}_n(A)$.

*Explicitly, for each object* $X_{(\phi_i)}$ *of* $\mathrm{Mor}_n(A)$, *the epimorphism*

$$\begin{pmatrix} (1,0,\cdots,0) \\ \vdots \\ (1,0) \\ 1 \end{pmatrix} : \ \mathrm{rMon}(X) \twoheadrightarrow X \tag{2.4}$$

*is a right minimal approximation of* $X$ *in* $\mathcal{S}_n(A)$.

**Proof.** By (2.2) and (2.3) it is easy to see that (2.4) is an epimorphism of $\mathrm{Mor}_n(A)$. Since $\mathcal{S}_n(A)$ is closed under subobjects, it follows that $\widehat{\mathcal{S}_n(A)} = \mathrm{Mor}_n(A)$. Let $\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} : Y_{(\psi_i)} \to X$ be a morphism of $\mathrm{Mor}_n(A)$ with $Y = Y_{(\psi_i)} \in \mathcal{S}_n(A)$. We need to find a morphism $g = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} : Y \to \mathrm{rMon}(X)$ such that

$$\begin{pmatrix} (1,0,\cdots,0) \\ \vdots \\ (1,0) \\ 1 \end{pmatrix} \begin{pmatrix} g_1 \\ \vdots \\ g_{n-1} \\ g_n \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_{n-1} \\ f_n \end{pmatrix}. \tag{2.5}$$

We will inductively construct $g_i = \begin{pmatrix} f_i \\ \alpha_{ii} \\ \vdots \\ \alpha_{in-1} \end{pmatrix} : Y_i \to X_i \oplus \mathrm{IKer}\,\phi_i \oplus \cdots \oplus \mathrm{IKer}\,\phi_{n-1},\ 1 \le i \le n-1,$ such that $g : Y \to \mathrm{rMon}(X)$ is a morphism of $\mathrm{Mor}_n(A)$, i.e., $\theta_i g_{i+1} = g_i \psi_i$, or explicitly, such that

$$\begin{pmatrix} \phi_i f_{i+1} \\ e_i f_{i+1} \\ \alpha_{i+1i+1} \\ \vdots \\ \alpha_{i+1n-1} \end{pmatrix} = \begin{pmatrix} f_i \psi_i \\ \alpha_{ii} \psi_i \\ \alpha_{ii+1} \psi_i \\ \vdots \\ \alpha_{in-1} \psi_i \end{pmatrix}. \tag{2.6}$$

Clearly $g_n = f_n$. Since $\psi_{n-1} : Y_n \hookrightarrow Y_{n-1}$ is monic and $\mathrm{IKer}\,\phi_{n-1}$ is an injective object, it follows that the composition $Y_n \xrightarrow{f_n} X_n \xrightarrow{e_{n-1}} \mathrm{IKer}\,\phi_{n-1}$ extends to a morphism $\alpha_{n-1n-1} : Y_{n-1} \to \mathrm{IKer}\,\phi_{n-1}$. Define $g_{n-1} = \begin{pmatrix} f_{n-1} \\ \alpha_{n-1\ n-1} \end{pmatrix}$. Then we have $\begin{pmatrix} \phi_{n-1} f_n \\ e_{n-1} f_n \end{pmatrix} = \begin{pmatrix} f_{n-1} \psi_{n-1} \\ \alpha_{n-1n-1} \psi_{n-1} \end{pmatrix}$. Assume that we have constructed $g_{n-1}, \cdots, g_t$ $(t \ge 2)$, such that (2.6) holds for $t \le i \le n-1$. Since

$\psi_{t-1}: Y_t \hookrightarrow Y_{t-1}$ is monic and $\operatorname{IKer}\phi_{t-1}$ is an injective object, it follows that the composition $Y_t \xrightarrow{f_t} X_t \xrightarrow{e_{t-1}} \operatorname{IKer}\phi_{t-1}$ extends to a morphism $\alpha_{t-1t-1}: Y_{t-1} \to \operatorname{IKer}\phi_{t-1}$. Similarly, for $t \le j \le n-1$, $\alpha_{tj}: Y_t \to \operatorname{IKer}\phi_j$ extends to a morphism $\alpha_{t-1j}: Y_{t-1} \to \operatorname{IKer}\phi_j$. Define

$$g_{t-1} = \begin{pmatrix} f_{t-1} \\ \alpha_{t-1t-1} \\ \alpha_{t-1t} \\ \vdots \\ \alpha_{t-1n-1} \end{pmatrix}.$$

By construction (2.6) holds for $i = t-1$, and then (2.5) is clearly satisfied. This proves that (2.4) is a right approximation of $X$ in $\mathcal{S}_n(A)$.

Now we prove that (2.4) is right minimal. Assume that $\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$ is an endomorphism of $\operatorname{rMon}(X)$ such that $\begin{pmatrix} (1,0,\cdots,0) & & \\ & \ddots & \\ & & (1,0) \\ & & & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_{n-1} \\ h_n \end{pmatrix} = \begin{pmatrix} (1,0,\cdots,0) \\ \vdots \\ (1,0) \\ 1 \end{pmatrix}$. We need to prove all $h_i$'s are isomorphisms. Write

$$h_{n-i+1}: \ X_{n-i+1} \oplus \operatorname{IKer}\phi_{n-i+1} \oplus \cdot \cdots \oplus \operatorname{IKer}\phi_{n-1} \longrightarrow X_{n-i+1} \oplus \operatorname{IKer}\phi_{n-i+1} \oplus \cdots \oplus \operatorname{IKer}\phi_{n-1}$$

as $\begin{pmatrix} h^{11}_{n-i+1} & \cdots & h^{1i}_{n-i+1} \\ \vdots & \cdots & \vdots \\ h^{i1}_{n-i+1} & \cdots & h^{ii}_{n-i+1} \end{pmatrix}$. Then $h_{n-i+1}$ is of the form $h_{n-i+1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ h^{21}_{n-i+1} & h^{22}_{n-i+1} & \cdots & h^{2i}_{n-i+1} \\ \vdots & \vdots & \cdots & \vdots \\ h^{i1}_{n-i+1} & h^{i2}_{n-i+1} & \cdots & h^{ii}_{n-i+1} \end{pmatrix}$. It suffices to prove that all $h_{n-i+1}$ are lower triangular matrices with diagonal elements being isomorphisms. We do this by induction. Clearly $h_n = 1$. From the commutative diagram

$$\begin{array}{ccc} X_n & \xrightarrow{\theta_{n-1}} & X_{n-1} \oplus \operatorname{IKer}\phi_{n-1} \\ \| & & \downarrow{\scriptstyle h_{n-1}} \\ X_n & \xrightarrow{\theta_{n-1}} & X_{n-1} \oplus \operatorname{IKer}\phi_{n-1} \end{array}$$

we have $\begin{pmatrix} 1 & 0 \\ h^{21}_{n-1} & h^{22}_{n-1} \end{pmatrix} \binom{\phi_{n-1}}{e_{n-1}} = \binom{\phi_{n-1}}{e_{n-1}}$, i.e., $h^{21}_{n-1}\phi_{n-1} + h^{22}_{n-1}e_{n-1} = e_{n-1}: X_n \to \operatorname{IKer}\phi_{n-1}$. Restricting the both sides to $\operatorname{Ker}\phi_{n-1}$ we get $h^{22}_{n-1}e'_{n-1} = e'_{n-1}: \operatorname{Ker}\phi_{n-1} \to \operatorname{IKer}\phi_{n-1}$ (see (2.1)). Since $e'_{n-1}$ is an injective envelope, by definition $h^{22}_{n-1}$ is an isomorphism.

Assume that $h_{n-t+1}$ $(t \ge 2)$ is a lower triangular matrix with diagonal elements being isomorphisms. Since $\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}: \ \operatorname{rMon}(X) \to \operatorname{rMon}(X)$ is a morphism, we have

$$\begin{pmatrix} \phi_{n-t} & 0 & 0 & \cdots & 0 \\ e_{n-t} & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{(t+1)\times t} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ h^{21}_{n-t+1} & h^{22}_{n-t+1} & \cdots & h^{2t}_{n-t+1} \\ \vdots & \vdots & \cdots & \vdots \\ h^{t1}_{n-t+1} & h^{t2}_{n-t+1} & \cdots & h^{tt}_{n-t+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ h^{21}_{n-t} & h^{22}_{n-t} & \cdots & h^{2t+1}_{n-t} \\ \vdots & \vdots & \cdots & \vdots \\ h^{t+11}_{n-t} & h^{t+12}_{n-t} & \cdots & h^{t+1t+1}_{n-t} \end{pmatrix} \begin{pmatrix} \phi_{n-t} & 0 & 0 & \cdots & 0 \\ e_{n-t} & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} \phi_{n-t} & 0 & 0 & \cdots & 0 \\ e_{n-t} & 0 & 0 & \cdots & 0 \\ h^{21}_{n-t+1} & h^{22}_{n-t+1} & h^{23}_{n-t+1} & \cdots & h^{2t}_{n-t+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ h^{t1}_{n-t+1} & h^{t2}_{n-t+1} & h^{t3}_{n-t+1} & \cdots & h^{tt}_{n-t+1} \end{pmatrix}_{(t+1)\times t} = \begin{pmatrix} \phi_{n-t} & 0 & 0 & \cdots & 0 \\ h^{21}_{n-t}\phi_{n-t}+h^{22}_{n-t}e_{n-t} & h^{23}_{n-t} & h^{24}_{n-t} & \cdots & h^{2t+1}_{n-t} \\ h^{31}_{n-t}\phi_{n-t}+h^{32}_{n-t}e_{n-t} & h^{33}_{n-t} & h^{34}_{n-t} & \cdots & h^{3t+1}_{n-t} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ h^{t+11}_{n-t}\phi_{n-t}+h^{t+12}_{n-t}e_{n-t} & h^{t+13}_{n-t} & h^{t+14}_{n-t} & \cdots & h^{t+1t+1}_{n-t} \end{pmatrix}_{(t+1)\times t}.$$

Comparing the second row of the both sides we see

$$h^{23}_{n-t} = 0,\ h^{24}_{n-t} = 0, \cdots, h^{2t+1}_{n-t} = 0. \tag{2.7}$$

For $3 \le i \le t+1,\ 2 \le j \le t$, comparing the $(i, j)$-entries in both sides we have

$$h^{ij+1}_{n-t} = h^{i-1j}_{n-t+1}. \tag{2.8}$$

Since $h^{ij}_{n-t+1} = 0$ for $j > i$ and $h^{22}_{n-t+1}, \cdots, h^{tt}_{n-t+1}$ are isomorphisms, it follows from (2.7) and (2.8) that $h^{ij}_{n-t} = h^{i-1j-1}_{n-t+1} = 0,\ \forall\ j > i$, and that $h^{ss}_{n-t} = h^{s-1s-1}_{n-t+1}$ for $s = 3, \cdots, t+1$. It remains to prove that $h^{22}_{n-t}$ is an isomorphism. Comparing the $(2, 1)$-entries we have

$$h^{21}_{n-t}\phi_{n-t} + h^{22}_{n-t}e_{n-t} = e_{n-t} :\ X_{n-t+1} \longrightarrow \mathrm{IKer}\,\phi_{n-t}.$$

Again restricting the both sides to $\mathrm{Ker}\,\phi_{n-t}$ and by a same argument we see that $h^{22}_{n-t}$ is an isomorphism. This completes the proof. ■

2.3. **Proof of Theorem 2.1.** By Corollary 1.4 and Lemma 2.3 $\mathcal{S}_n(A)$ is a resolving contravariantly finite subcategory in $T_n(A)$-mod. Then by Corollary 0.3 of [KS] (which asserts that a resolving contravariantly finite subcategory in $A$-mod is also covariantly finite in $A$-mod) $\mathcal{S}_n(A)$ is a functorially finite subcategory in $T_n(A)$-mod. Thus $\mathcal{S}_n(A)$ has Auslander-Reiten sequences, by Theorem 2.4 of [AS]. ■

2.4. For a later use we write down the dual of Theorem 2.1.

**Theorem 2.1'.** *Let $A$ be an Artin algebra. Then $\mathcal{F}_n(A)$ is a functorially finite subcategory in* $\mathrm{Mor}_n(A)$ *and has Auslander-Reiten sequences.*

## 3. Monomorphism categories and cotilting theory

The promised reciprocity will be proved, and some consequences will be given.

3.1. Let $D$ be the duality $A^{op}$-mod $\to$ $A$-mod. For $M \in A$-mod, denote by $\mathrm{add}(M)$ the full subcategory of $A$-mod consisting of all the direct summands of finite direct sums of copies of $M$, and by ${}^{\perp}M$ the full subcategory of $A$-mod given by $\{X \in A\text{-mod} \mid \mathrm{Ext}^i_A(X, M) = 0,\ \forall\ i \ge 1\}$.

An $A$-module $T$ is *an $r$-cotilting module* if the following three conditions are satisfied

$(i)$ inj.dim$T \le r$;
$(ii)$ $\mathrm{Ext}^i_A(T, T) = 0$ for $i \ge 1$; and
$(iii)$ there is an exact sequence $0 \to T_s \to \cdots \to T_0 \to D(A_A) \to 0$ with $T_i \in \mathrm{add}(T),\ 0 \le i \le s$.

We refer to [HR] and [AR] for the tilting theory.

Given an $A$-module $M$, using functor $\mathbf{m}_i: A\text{-mod} \to T_n(A)\text{-mod}$ we have a $T_n(A)$-module

$$\mathbf{m}(M) = \bigoplus_{1\le i\le n} \mathbf{m}_i(M) = \begin{pmatrix} M \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \oplus \begin{pmatrix} M \\ M \\ 0 \\ \vdots \\ 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} M \\ M \\ M \\ \vdots \\ M \end{pmatrix}. \tag{3.1}$$

The key result of this paper is as follows.

**Theorem 3.1.** *Let $A$ be an Artin algebra, and $T$ an $A$-module.*

($i$) *If there is an exact sequence* $0 \to T_s \to \cdots \to T_0 \to D(A_A) \to 0$ *with* $T_i \in \operatorname{add}(T),\ 0 \le i \le s$, *then* $\mathcal{S}_n(^\perp T) = {}^\perp\mathbf{m}(T)$.

($ii$) *If $T$ is a cotilting $A$-module, then $\mathbf{m}(T)$ is a unique cotilting $T_n(A)$-module, up to multiplicities of indecomposable direct summands, such that* $\mathcal{S}_n(^\perp T) = {}^\perp\mathbf{m}(T)$.

Taking $T = D(A_A)$ in Theorem 3.1($ii$) we have

**Corollary 3.2.** *Let $A$ be an Artin algebra. Then* $\mathcal{S}_n(A) = {}^\perp\mathbf{m}(D(A_A))$.

In fact, $\mathbf{m}(D(A_A))$ is the unique cotilting $T_n(A)$-module, up to multiplicities of indecomposable direct summands, such that $\mathcal{S}_n(A) = {}^\perp\mathbf{m}(D(A_A))$; moreover, inj.dim $\mathbf{m}(D(A_A)) = 1$, and $\operatorname{End}_{T_n(A)}(\mathbf{m}(D(A_A))) \cong (T_n(A))^{op}$. Note that the unique existence of a cotilting $T_n(A)$-module $C$ such that $\mathcal{S}_n(A) = {}^\perp C$ is also guaranteed by Theorem 5.5(a) in [AR]: since by Lemma 2.3 and Corollary 1.4 $\mathcal{S}_n(A)$ is a resolving contravariantly finite subcategory in $T_n(A)$-mod with $\widehat{\mathcal{S}_n(A)} = T_n(A)$-mod.

If inj.dim$A_A < \infty$, we can take $T = {}_AA$ in Theorem 3.1($i$) to get

**Corollary 3.3.** *Let $A$ be an Artin algebra with* inj.dim$A_A < \infty$. *Then* $\mathcal{S}_n(^\perp A) = {}^\perp\mathbf{m}(A)$.

3.2. The proof of Theorem 3.1 needs Theorem 2.1 and the six adjoint pairs between $A$-mod and $T_n(A)$-mod, which were implied by Lemma 1.2 and will be further explored in the following.

**Lemma 3.4.** *Let $A$ be an Artin algebra and $M$ an arbitrary $A$-module. Then*

($i$) *For each $X \in T_n(A)$-mod, we have isomorphisms of abelian groups, which are natural in both positions*

$$\operatorname{Ext}^j_{T_n(A)}(\mathbf{m}_i(M), X) \cong \operatorname{Ext}^j_A(M, X_i), \quad j \ge 0,\ 1 \le i \le n, \tag{3.2}$$

$$\operatorname{Ext}^j_{T_n(A)}(X, \mathbf{m}_n(M)) \cong \operatorname{Ext}^j_A(X_1, M), \quad j \ge 0, \tag{3.3}$$

$$\operatorname{Ext}^j_{T_n(A)}(X, \mathbf{p}_i(M)) \cong \operatorname{Ext}^j_A(X_{n-i+1}, M), \quad j \ge 0,\ 1 \le i \le n, \tag{3.4}$$

$$\operatorname{Ext}^j_{T_n(A)}(\mathbf{p}_n(M), X) \cong \operatorname{Ext}^j_A(M, X_n),\ j \ge 0. \tag{3.5}$$

($ii$) *For each $X = X_{(\phi_i)} \in \mathcal{S}_n(A)$, we have isomorphisms of abelian groups, which are natural in both positions*

$$\operatorname{Ext}^j_{T_n(A)}(X, \mathbf{m}_i(M)) \cong \operatorname{Ext}^j_A(\operatorname{Coker}(\phi_1\cdots\phi_i), M), \quad j \ge 0,\ 1 \le i \le n-1. \tag{3.6}$$

*$(ii)'$ For each $X = X_{(\phi_i)} \in \mathcal{F}_n(A)$, we have isomorphisms of abelian groups, which are natural in both positions*

$$\mathrm{Ext}^j_{T_n(A)}(\mathbf{p}_i(M), X) \cong \mathrm{Ext}^j_A(M, \mathrm{Ker}(\phi_{n-i}\cdots\phi_{n-1})), \quad j \geq 0,\ 1 \leq i \leq n-1. \tag{3.7}$$

**Proof.** $(i)$ We justify (3.3). Taking the 1-st branch of a projective resolution

$$\cdots \longrightarrow P^1_{(\psi^1_i)} \longrightarrow P^0_{(\psi^0_i)} \longrightarrow X_{(\phi_i)} \longrightarrow 0 \tag{$*$}$$

of $X = X_{(\phi_i)}$, by (1.8) we get a projective resolution $\cdots \to P^1_1 \to P^0_1 \to X_1 \to 0$ of $X_1$. On the other hand, by (1.4) we get the following isomorphic complexes (for saving the space I omit Hom, and same convention below)

$$\begin{tikzcd}
0 \arrow[r] & (X, \mathbf{m}_n(M)) \arrow[r] \arrow[d, "\wr"] & (P^0, \mathbf{m}_n(M)) \arrow[r] \arrow[d, "\wr"] & (P^1, \mathbf{m}_n(M)) \arrow[r] \arrow[d, "\wr"] & \cdots \\
0 \arrow[r] & (X_1, M) \arrow[r] & (P^0_1, M) \arrow[r] & (P^1_1, M) \arrow[r] & \cdots .
\end{tikzcd}$$

This implies (3.3).

$(ii)$ By Corollary 1.4 $\mathcal{S}_n(A)$ is a resolving subcategory of $T_n(A)$-mod, hence by Lemma 1.1$(i)$ we deduce from $(*)$ that

$$\cdots \longrightarrow \mathrm{Coker}(\psi^1_1\cdots\psi^1_i) \longrightarrow \mathrm{Coker}(\psi^0_1\cdots\psi^0_i) \longrightarrow \mathrm{Coker}(\phi_1\cdots\phi_i) \longrightarrow 0$$

is also exact (it is here we need the assumption $X_{(\phi_i)} \in \mathcal{S}_n(A)$). By (1.8) we see that $\mathrm{Coker}(\psi^j_1\cdots\psi^j_i)$ is again a projective $A$-module for every $j$ (it suffices to see this for indecomposable projective $T_n(A)$-modules, which is of the form $\mathbf{m}_i(P)$), it follows that this exact sequence turns out to be a projective resolution of $\mathrm{Coker}(\phi_1\cdots\phi_i)$. On the other hand, for each $1 \leq i \leq n-1$, by (1.3) we get the following two isomorphic complexes

$$\begin{tikzcd}
0 \arrow[r] & (X, \mathbf{m}_i(M)) \arrow[r] \arrow[d, "\wr"] & (P^0, \mathbf{m}_i(M)) \arrow[r] \arrow[d, "\wr"] & (P^1, \mathbf{m}_i(M)) \arrow[r] \arrow[d, "\wr"] & \cdots \\
0 \arrow[r] & (\mathrm{Coker}(\phi_1\cdots\phi_i), M) \arrow[r] & (\mathrm{Coker}(\psi^0_1\cdots\psi^0_i), M) \arrow[r] & (\mathrm{Coker}(\psi^1_1\cdots\psi^1_i), M) \arrow[r] & \cdots .
\end{tikzcd}$$

This implies (3.6). ■

3.3. The proof of the following lemma needs Ringel - Schmidmeier's theorem.

**Lemma 3.5.** *Let $A$ be an Artin algebra and $X_{(\phi_i)}$ a $T_n(A)$-module. If $X_{(\phi_i)} \in {}^{\perp}\mathbf{m}(D(A_A))$, then $\phi_i$ is monic, $1 \leq j \leq n-1$.*

**Proof.** Taking a right minimal approximation of $X_{(\phi_i)}$ in $\mathcal{S}_n(A)$, by (2.4) we have an exact sequence

$$0 \longrightarrow K_{(\theta'_i)} \longrightarrow (\mathrm{rMon}(X))_{(\theta_i)} \longrightarrow X_{(\phi_i)} \longrightarrow 0. \tag{$**$}$$

Applying $\mathrm{Hom}_{T_n(A)}(-, \mathbf{m}_i(D(A_A)))$ to $(**)$ we get an exact sequence, and by (1.3) this exact sequence is

$$0 \longrightarrow (\mathrm{Coker}(\phi_1\cdots\phi_i), D(A_A)) \longrightarrow (\mathrm{Coker}(\theta_1\cdots\theta_i), D(A_A)) \longrightarrow (\mathrm{Coker}(\theta'_1\cdots\theta'_i), D(A_A)) \longrightarrow 0,$$

and hence we get the following exact sequence, which is induced by $(**)$

$$0 \longrightarrow \operatorname{Coker}(\theta'_1 \cdots \theta'_i) \longrightarrow \operatorname{Coker}(\theta_1 \cdots \theta_i) \longrightarrow \operatorname{Coker}(\phi_1 \cdots \phi_i) \longrightarrow 0.$$

On the other hand, since $\theta_1 \cdots \theta_i$ is monic, by Lemma 1.1$(i)$ we get the following exact sequence, which is again induced by $(**)$

$$0 \longrightarrow \operatorname{Ker}(\phi_1 \cdots \phi_i) \longrightarrow \operatorname{Coker}(\theta'_1 \cdots \theta'_i) \longrightarrow \operatorname{Coker}(\theta_1 \cdots \theta_i) \longrightarrow \operatorname{Coker}(\phi_1 \cdots \phi_i) \longrightarrow 0.$$

Thus $\operatorname{Ker}(\phi_1 \cdots \phi_i) = 0$, and hence $\phi_i$ is monic for $1 \le i \le n-1$. ■

Given an $A$-module $M$, using functor $\mathbf{p}_i : A\text{-mod} \to T_n(A)\text{-mod}$ we get a $T_n(A)$-module

$$\mathbf{p}(M) = \bigoplus_{1 \le i \le n} \mathbf{p}_i(M) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ M \end{pmatrix} \oplus \begin{pmatrix} 0 \\ \vdots \\ 0 \\ M \\ M \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} M \\ M \\ M \\ \vdots \\ M \end{pmatrix}.$$

**Proposition 3.6.** *Let $A$ be an Artin algebra and $T$ an arbitrary $A$-module. Then*

$$\mathcal{S}_n({}^{\perp}T) = {}^{\perp}\mathbf{m}(T) \cap {}^{\perp}\mathbf{p}(T) \cap {}^{\perp}\mathbf{m}(D(A_A)).$$

**Proof.** By (3.4) we have $\mathcal{S}_n({}^{\perp}T) \subseteq {}^{\perp}\mathbf{p}(T)$. By (3.3) and (3.6) we have $\mathcal{S}_n({}^{\perp}T) \subseteq {}^{\perp}\mathbf{m}(D(A_A))$. Let $X_{(\phi_i)} \in \mathcal{S}_n({}^{\perp}T)$. By definition $\phi_i : X_{i+1} \hookrightarrow X_i$ is monic and $\operatorname{Coker}\phi_i \in {}^{\perp}T,\ 1 \le i \le n-1$. By the exact sequence $0 \to \operatorname{Coker}\phi_i \to \operatorname{Coker}(\phi_1 \cdots \phi_i) \to \operatorname{Coker}(\phi_1 \cdots \phi_{i-1}) \to 0$ we inductively see $\operatorname{Coker}(\phi_1 \cdots \phi_i) \in {}^{\perp}T$ for $1 \le i \le n-1$, and then by (3.6) and (3.3) this means $X_{(\phi_i)} \in {}^{\perp}\mathbf{m}(T)$. This proves $\mathcal{S}_n({}^{\perp}T) \subseteq {}^{\perp}\mathbf{m}(T) \cap {}^{\perp}\mathbf{p}(T) \cap {}^{\perp}\mathbf{m}(D(A_A))$.

Conversely, let $X_{(\phi_i)} \in {}^{\perp}\mathbf{m}(T) \cap {}^{\perp}\mathbf{p}(T) \cap {}^{\perp}\mathbf{m}(D(A_A))$. Then by (3.4) we have $X_i \in {}^{\perp}T,\ 1 \le i \le n$, and by Lemma 3.5 $\phi_i : X_{i+1} \hookrightarrow X_i$ is monic, $1 \le i \le n-1$. By (3.6) we know $\operatorname{Coker}(\phi_1 \cdots \phi_i) \in {}^{\perp}T$ for $1 \le i \le n-1$, and from the exact sequence $0 \to \operatorname{Coker}\phi_i \to \operatorname{Coker}(\phi_1 \cdots \phi_i) \to \operatorname{Coker}(\phi_1 \cdots \phi_{i-1}) \to 0$ we know $\operatorname{Coker}\phi_i \in {}^{\perp}T,\ 1 \le i \le n-1$. This proves $X_{(\phi_i)} \in \mathcal{S}_n({}^{\perp}T)$ and completes the proof. ■

3.4. Now we deal with cotilting modules.

**Lemma 3.7.** *Let $A$ be an Artin algebra and $T$ an $r$-cotilting $A$-module. Then $\mathbf{m}(T)$ is an $(r+1)$-cotilting $T_n(A)$-module with $\operatorname{End}_{T_n(A)}(\mathbf{m}(T)) \cong (T_n(\operatorname{End}_A(T)))^{op}$.*

**Proof.** By (1.2) we have

$$\operatorname{Hom}_{T_n(A)}(\mathbf{m}_i(T), \mathbf{m}_j(T)) \cong \begin{cases} 0, & i > j, \\ \operatorname{End}_A(T), & i \le j, \end{cases}$$

we infer that $\operatorname{End}_{T_n(A)}(\mathbf{m}(T)) \cong (T_n(\operatorname{End}_A(T)))^{op}$.

Assume that $\text{inj.dim}T = r$ with a minimal injective resolution $0 \to T \to I_0 \to \cdots \to I_r \to 0$. Since $\mathbf{m}_n : A\text{-mod} \to T_n(A)\text{-mod}$ is an exact functor and $\mathbf{m}_n(I_j) = \mathbf{p}_n(I_j)$ is an injective $T_n(A)$-module for each $j$, it follows that $\text{inj.dim}\ \mathbf{m}_n(T) = r$. Similarly, $\text{inj.dim}\ \mathbf{p}_{n-i}(T) = r$ for

$1 \le i \le n-1$, and then by the following exact sequence

$$0 \longrightarrow \mathbf{m}_i(T) = \begin{pmatrix} T \\ \vdots \\ T \\ 0 \\ \vdots \\ 0 \end{pmatrix} \longrightarrow \mathbf{m}_n(T) = \begin{pmatrix} T \\ \vdots \\ T \\ T \\ \vdots \\ T \end{pmatrix} \longrightarrow \mathbf{p}_{n-i}(T) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ T \\ \vdots \\ T \end{pmatrix} \longrightarrow 0$$

we see inj.dim $\mathbf{m}_i(T) \le r+1$. Thus inj.dim $\mathbf{m}(T) \le r+1$.

By (3.2) we have

$$\mathrm{Ext}^j_{T_n(A)}(\mathbf{m}_i(T), \mathbf{m}(T)) = \mathrm{Ext}^j_A(T, (\mathbf{m}(T))_i) = \mathrm{Ext}^j_A(T, \underbrace{T \oplus \cdots \oplus T}_{n-i+1}) = 0,\ j \ge 1,\ 1 \le i \le n.$$

This proves $\mathrm{Ext}^j_{T_n(A)}(\mathbf{m}(T), \mathbf{m}(T)) = 0$ for $j \ge 1$.

Since $T$ is a cotilting $A$-module, we have an exact sequence

$$0 \longrightarrow T_s \longrightarrow T_{s-1} \longrightarrow \cdots \longrightarrow T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{d_0} D(A_A) \longrightarrow 0$$

with every $T_j \in \mathrm{add}(T)$. Clearly we have an exact sequence

$$0 \longrightarrow \mathbf{m}_n(T_s) \longrightarrow \cdots \longrightarrow \mathbf{m}_n(T_1) \xrightarrow{\mathbf{m}_n(d_1)} \mathbf{m}_n(T_0) \xrightarrow{\mathbf{m}_n(d_0)} \mathbf{m}_n(D(A_A)) = \mathbf{p}_n(D(A_A)) \longrightarrow 0$$

with every $\mathbf{m}_n(T_j) \in \mathrm{add}(\mathbf{m}_n(T)) \subseteq \mathrm{add}(\mathbf{m}(T))$. For $1 \le i \le n-1$, we have the following exact sequence of $T_n(A)$-modules

$$0 \longrightarrow \begin{pmatrix} T_0 \\ \vdots \\ T_0 \\ \mathrm{Ker}\, d_0 \\ \vdots \\ \mathrm{Ker}\, d_0 \end{pmatrix}_{(\phi^0_j)} \xrightarrow{\begin{pmatrix} 1 \\ \vdots \\ 1 \\ a \\ \vdots \\ a \end{pmatrix}} \mathbf{m}_n(T_0) = \begin{pmatrix} T_0 \\ \vdots \\ T_0 \\ T_0 \\ \vdots \\ T_0 \end{pmatrix} \xrightarrow{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ d_0 \\ \vdots \\ d_0 \end{pmatrix}} \mathbf{p}_i(D(A_A)) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ D(A) \\ \vdots \\ D(A) \end{pmatrix} \longrightarrow 0,$$

where

$$\phi^0_j = \begin{cases} \mathrm{id}_{T_0}, & 1 \le j \le n-i-1, \\ a, & j = n-i, \\ \mathrm{id}_{\mathrm{Ker}\, d_0}, & n-i+1 \le j \le n-1, \end{cases}$$

and $a : \mathrm{Ker}\, d_0 \hookrightarrow T_0$ is the embedding. Consider the following sequence of $T_n(A)$-modules

$$0 \longrightarrow \begin{pmatrix} T_1 \\ \vdots \\ T_1 \\ \mathrm{Ker}\, d_1 \\ \vdots \\ \mathrm{Ker}\, d_1 \end{pmatrix}_{(\phi^1_j)} \xrightarrow{f} \begin{pmatrix} T_1 \\ \vdots \\ T_1 \\ T_1 \\ \vdots \\ T_1 \end{pmatrix} \oplus \begin{pmatrix} T_0 \\ \vdots \\ T_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \xrightarrow{\left(\begin{pmatrix} d_1 \\ \vdots \\ d_1 \\ d_1 \\ \vdots \\ d_1 \end{pmatrix}, \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right)} \begin{pmatrix} T_0 \\ \vdots \\ T_0 \\ \mathrm{Ker}\, d_0 \\ \vdots \\ \mathrm{Ker}\, d_0 \end{pmatrix}_{(\phi^0_j)} \longrightarrow 0, \qquad (3.8)$$

where

$$f=\begin{pmatrix}\binom{1}{-d_1}\\ \vdots\\ \binom{1}{-d_1}\\ a\\ \vdots\\ a\end{pmatrix}:\ \begin{pmatrix}T_1\\ \vdots\\ T_1\\ \operatorname{Ker} d_1\\ \vdots\\ \operatorname{Ker} d_1\end{pmatrix}\longrightarrow\begin{pmatrix}T_1\oplus T_0\\ \vdots\\ T_1\oplus T_0\\ T_1\\ \vdots\\ T_1\end{pmatrix}.$$

It is routine to see that $f$ and all other maps in (3.8) are $T_n(A)$-maps (i.e., (1.1) is satisfied for each map). Since both $0\to T_1\xrightarrow{\binom{1}{-d_1}}T_1\oplus T_0\xrightarrow{(d_1,1)}T_0\to 0$ and $0\to\operatorname{Ker} d_1\overset{a}{\hookrightarrow}T_1\xrightarrow{d_1}\operatorname{Ker} d_0\to 0$ are exact, it follows that (3.8) is exact.

Repeating this process we get the following exact sequence of $T_n(A)$-modules

$$\begin{aligned}0\longrightarrow \mathbf{m}_{n-i}(T_s)\longrightarrow \mathbf{m}_n(T_s)\oplus\mathbf{m}_{n-i}(T_{s-1})\longrightarrow\cdots\longrightarrow\mathbf{m}_n(T_2)\oplus\mathbf{m}_{n-i}(T_1)\\ \longrightarrow\mathbf{m}_n(T_1)\oplus\mathbf{m}_{n-i}(T_0)\longrightarrow\mathbf{m}_n(T_0)\longrightarrow\mathbf{p}_i(D(A_A))\longrightarrow 0.\end{aligned}$$

By definition $\mathbf{m}(T)$ is a cotilting $T_n(A)$-module. ■

3.5. **Proof of Theorem 3.1.** $(i)$ By Proposition 3.6 we have $\mathcal{S}_n({}^{\perp}T)\subseteq{}^{\perp}\mathbf{m}(T)$. Let $X=X_{(\phi_i)}\in{}^{\perp}\mathbf{m}(T)$. By the assumption on $T$ we have an exact sequence

$$0\longrightarrow\mathbf{m}_i(T_s)\longrightarrow\cdots\longrightarrow\mathbf{m}_i(T_0)\longrightarrow\mathbf{m}_i(D(A_A))\longrightarrow 0,\quad 1\le i\le n,$$

with $\mathbf{m}_i(T_j)\in\operatorname{add}(\mathbf{m}_i(T))$, $0\le j\le s$. It follows from $X_{(\phi_j)}\in{}^{\perp}\mathbf{m}_i(T)$ that $X_{(\phi_j)}\in{}^{\perp}\mathbf{m}_i(D(A_A))$, and hence by Lemma 3.5 $\phi_i$ is monic for $1\le i\le n-1$. Now we can use (3.6) to get $\operatorname{Coker}(\phi_1\cdots\phi_i)\in{}^{\perp}T$ for $1\le i\le n-1$, and by (3.3) we have $X_1\in{}^{\perp}T$. From the exact sequence $0\to X_{i+1}\overset{\phi_1\cdots\phi_i}{\hookrightarrow}X_1\twoheadrightarrow\operatorname{Coker}(\phi_1\cdots\phi_i)\to 0$ we know $X_{i+1}\in{}^{\perp}T$, $1\le i\le n-1$. From the exact sequence $0\to\operatorname{Coker}\phi_i\to\operatorname{Coker}(\phi_1\cdots\phi_i)\to\operatorname{Coker}(\phi_1\cdots\phi_{i-1})\to 0$ we know $\operatorname{Coker}\phi_i\in{}^{\perp}T$, $1\le i\le n-1$. This proves $X_{(\phi_i)}\in\mathcal{S}_n({}^{\perp}T)$ and hence $\mathcal{S}_n({}^{\perp}T)={}^{\perp}\mathbf{m}(T)$.

$(ii)$ It follows from $(i)$ and Lemma 3.7 that $\mathbf{m}(T)$ is a cotilting $T_n(A)$-module with $S_n({}^{\perp}T)={}^{\perp}\mathbf{m}(T)$. The remaining uniqueness follows from D. Happel's result on the number of pairwise non-isomorphic direct summands of a cotilting module ([H1]). This completes the proof. ■

3.6. With the similar arguments one can prove

**Proposition 3.8.** *Let $A$ be an Artin algebra and $T$ an arbitrary $A$-module. Then*

$$\mathcal{S}_n(T^{\perp})=\mathbf{m}(T)^{\perp}\cap{}^{\perp}\mathbf{m}(D(A_A)).$$

*Moreover, if there is an exact sequence $0\to T_s\to\cdots\to T_0\to D(A_A)\to 0$ with $T_i\in\operatorname{add}(T)$, $0\le i\le s$, then $\mathcal{S}_n({}^{\perp}T\cap T^{\perp})={}^{\perp}\mathbf{m}(T)\cap\mathbf{m}(T)^{\perp}$.*

As an application of Theorem 3.1, we get an answer to the main problem in Introduction.

**Theorem 3.9.** *Let $A$ be an Artin algebra and $\mathcal{X}$ a full subcategory of $A$-mod. Then $\mathcal{S}_n(\mathcal{X})$ is a resolving contravariantly finite subcategory in $T_n(A)$-mod with $\widehat{\mathcal{S}_n(\mathcal{X})}=T_n(A)$-mod if and only if $\mathcal{X}$ is a resolving contravariantly finite subcategory in $A$-mod with $\widehat{\mathcal{X}}=A$-mod.*

**Proof.** If $\mathcal{X}$ is a resolving contravariantly finite subcategory in $A$-mod with $\widehat{\mathcal{X}} = A$-mod, then by Theorem 5.5(a) of Auslander-Reiten [AR] there is a cotilting $A$-module $T$ such that $\mathcal{X} = {}^{\perp}T$. By Theorem 3.1$(ii)$ $\mathbf{m}(T)$ is a cotilting $T_n(A)$-module such that $\mathcal{S}_n(\mathcal{X}) = {}^{\perp}\mathbf{m}(T)$. Again by Theorem 5.5(a) in [AR] we know that $\mathcal{S}_n(\mathcal{X})$ is a resolving contravariantly finite subcategory in $T_n(A)$-mod with $\widehat{\mathcal{S}_n(\mathcal{X})} = T_n(A)$-mod. Conversely, assume that $\mathcal{S}_n(\mathcal{X})$ is a resolving contravariantly finite subcategory in $T_n(A)$-mod with $\widehat{\mathcal{S}_n(\mathcal{X})} = T_n(A)$-mod. By Corollary 1.4 $\mathcal{X}$ is a resolving subcategory of $A$-mod. Since $\mathcal{S}_n(\mathcal{X})$ is contravariantly finite in $T_n(A)$-mod with $\widehat{\mathcal{S}_n(\mathcal{X})} = T_n(A)$-mod, by using functor $\mathbf{m}_1 : A\text{-mod} \to \mathcal{S}_n(A)$, which induces a functor $\mathbf{m}_1 : \mathcal{X} \to \mathcal{S}_n(\mathcal{X})$, we infer that $\mathcal{X}$ is contravariantly finite subcategory in $A$-mod with $\widehat{\mathcal{X}} = A$-mod. ■

3.7. For a later use we write down the dual versions of Theorem 3.1, Corollaries 3.2 and 3.3, Propositions 3.6 and 3.8.

**Theorem 3.1'.** *Let $A$ be an Artin algebra and $T$ an arbitrary $A$-module.*

$(i)$ *We have* $\mathcal{F}_n(T^{\perp}) = \mathbf{p}(T)^{\perp} \cap \mathbf{m}(T)^{\perp} \cap \mathbf{p}(A)^{\perp}$.

$(ii)$ *If there is an exact sequence* $0 \to A \to T_0 \to \cdots \to T_s \to 0$ *with* $T_i \in \operatorname{add}(T)$, $0 \le i \le s$, *then* $\mathcal{F}_n(T^{\perp}) = \mathbf{p}(T)^{\perp}$.

$(iii)$ *If $T$ is a tilting $A$-module, then* $\mathbf{p}(T)$ *is a unique tilting $T_n(A)$-module, up to multiplicities of indecomposable direct summands, such that* $\mathcal{F}_n(T^{\perp}) = \mathbf{p}(T)^{\perp}$.

$(iv)$ $\mathbf{p}(A)$ *is the unique tilting $T_n(A)$-module, up to multiplicities of indecomposable direct summands, such that* $\mathcal{F}_n(A) = \mathbf{p}(A)^{\perp}$. *Moreover,* proj.dim $\mathbf{p}(A) = 1$, *and* $\operatorname{End}_{T_n(A)}(\mathbf{p}(A)) \cong (T_n(A))^{op}$.

$(v)$ *If* $\text{inj.dim}_A A < \infty$, *then* $\mathcal{F}_n(D(A_A)^{\perp}) = \mathbf{p}(D(A_A))^{\perp}$.

$(vi)$ *We have* $\mathcal{F}_n({}^{\perp}T) = {}^{\perp}\mathbf{p}(T) \cap \mathbf{p}(A)^{\perp}$.

$(vii)$ *If there is an exact sequence* $0 \to A \to T_0 \to \cdots \to T_s \to 0$ *with* $T_i \in \operatorname{add}(T)$, $0 \le i \le s$, *then* $\mathcal{F}_n(T^{\perp} \cap {}^{\perp}T) = \mathbf{p}(T)^{\perp} \cap {}^{\perp}\mathbf{p}(T)$.

3.8. We have the following

**Remark 3.10.** $(i)$ *The converse of Theorem 3.1$(i)$ is* **not** *true. For example, let $k$ be a field and $A$ be the path $k$-algebra of the quiver* $1\bullet \longrightarrow 2\bullet$. *Then $T_2(A)$ is the algebra given by the quiver*

$$\begin{tikzcd} & \bullet \arrow[dr, "\beta"] & \\ \bullet \arrow[ur, "\alpha"] \arrow[dr, "\gamma"'] & & \bullet \\ & \bullet \arrow[ur, "\delta"'] & \end{tikzcd}$$

*with relation $\beta\alpha - \delta\gamma$. The Auslander-Reiten quiver of $T_2(A)$ is*

$$\begin{tikzcd}[column sep=small]
 & \left(\begin{smallmatrix} P(1) \\ 0 \end{smallmatrix}\right) \arrow[dr] & & \left(\begin{smallmatrix} 0 \\ S(2) \end{smallmatrix}\right) \arrow[dr] & & \left(\begin{smallmatrix} S(1) \\ S(1) \end{smallmatrix}\right) \arrow[dr] & \\
\left(\begin{smallmatrix} S(2) \\ 0 \end{smallmatrix}\right) \arrow[ur] \arrow[dr] & & \left(\begin{smallmatrix} P(1) \\ S(2) \end{smallmatrix}\right)_\sigma \arrow[ur] \arrow[r] \arrow[dr] & \left(\begin{smallmatrix} P(1) \\ P(1) \end{smallmatrix}\right) \arrow[r] & \left(\begin{smallmatrix} S(1) \\ P(1) \end{smallmatrix}\right)_p \arrow[ur] \arrow[dr] & & \left(\begin{smallmatrix} 0 \\ S(1) \end{smallmatrix}\right) \\
 & \left(\begin{smallmatrix} S(2) \\ S(2) \end{smallmatrix}\right) \arrow[ur] & & \left(\begin{smallmatrix} S(1) \\ 0 \end{smallmatrix}\right) \arrow[ur] & & \left(\begin{smallmatrix} 0 \\ P(1) \end{smallmatrix}\right) \arrow[ur] &
\end{tikzcd}$$

*Let $T = S(1) \oplus S(2)$. Then $\mathcal{S}_2(^\perp T) = \mathcal{S}_2(^\perp S(2)) = \mathcal{S}_2(\mathrm{add}(A))$, and*

$$^\perp\mathbf{m}(T) = \ ^\perp(\left(\begin{smallmatrix} S(1)\oplus S(2) \\ 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} S(1)\oplus S(2) \\ S(1)\oplus S(2) \end{smallmatrix}\right)) = \ ^\perp\left(\begin{smallmatrix} S(1) \\ 0 \end{smallmatrix}\right) \cap \ ^\perp\left(\begin{smallmatrix} S(2) \\ 0 \end{smallmatrix}\right) \cap \ ^\perp\left(\begin{smallmatrix} S(2) \\ S(2) \end{smallmatrix}\right).$$

*It is clear that*

$$\mathcal{S}_2(^\perp T) = \mathcal{S}_2(\mathrm{add}(A)) = \mathrm{add}(\left(\begin{smallmatrix} S(2) \\ 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} P(1) \\ 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} S(2) \\ S(2) \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} P(1) \\ P(1) \end{smallmatrix}\right)) = \ ^\perp\mathbf{m}(T),$$

*but $T$ does not satisfy the condition in Theorem 3.1(i).*

*Nevertheless, even in this example, for many $A$-modules $T$ not satisfying the condition in Theorem 3.1(i), we have $\mathcal{S}_n(^\perp T) \neq \ ^\perp\mathbf{m}(T)$. For examples, this is the case when $T = S(1)$, or $T = S(2)$, or $T = P(1)$.*

*(ii) Many cotilting $T_n(A)$-modules are not of the form $\mathbf{m}(T)$, where $T$ is a cotilting $A$-module. For example, if $k$ is a field, then $T_3(k)$ is the path $k$-algebra of the quiver $1\bullet \longrightarrow 2\bullet \longrightarrow 3\bullet$. There two basic cotilting $T_3(k)$-modules having the simple module $S(2)$ as a direct summand, which are not of the form $\mathbf{m}(T)$, where $T \in k$-mod.*

## 4. **Application to Gorenstein algebras**

Applying Theorem 3.1 to Gorenstein algebras, we explicitly determine all the Gorenstein-projective $T_n(A)$-modules. We characterize self-injective algebras by monomorphism categories.

4.1. Modules in $^\perp(_AA)$ are called *Cohen-Macaulay $A$-modules*. Denote $^\perp A$ by $\mathbf{CM}(A)$. An $A$-module $G$ is *Gorenstein-projective*, if there is an exact sequence $\cdots \to P^{-1} \to P^0 \stackrel{d^0}{\to} P^1 \to \cdots$ of projective $A$-modules, which stays exact under $\mathrm{Hom}_A(-, A)$, and such that $G \cong \mathrm{Ker}\, d^0$. Let $A$-$\mathcal{G}proj$ be the full subcategory of $A$-mod of Gorenstein-projective modules. Then $A$-$\mathcal{G}proj \subseteq \mathbf{CM}(A)$; and if $A$ is *a Gorenstein algebra* (i.e., inj.dim $_AA < \infty$ and inj.dim $A_A < \infty$), then $A$-$\mathcal{G}proj = \mathbf{CM}(A)$ (Enochs - Jenda [EJ2], Corollary 11.5.3). Determining all the Cohen-Macaulay $A$-modules and all the Gorenstein-projective $A$-modules in explicit way, is a basic requirement in applications (see e.g. [AM], [B], [BGS], [CPST], [EJ2], [GZ], [K]).

**Corollary 4.1.** *(i) Let $A$ be an Artin algebra with* inj.dim$A_A < \infty$. *Then*

$$\mathbf{CM}(T_n(A)) = \mathcal{S}_n(\mathbf{CM}(A)).$$

*(ii) Let $A$ be a Gorenstein algebra. Then $T_n(A)$-$\mathcal{G}proj = \mathcal{S}_n(A$-$\mathcal{G}proj)$.*

**Proof.** $(i)$ is a reformulation of Corollary 3.3 since $\mathbf{m}({}_AA) = {}_{T_n(A)}T_n(A)$. If $A$ is Gorenstein, then it is well-known that $T_n(A)$ is again Gorenstein (for $n = 2$ see e.g. [FGR] or [H2]; in general see e.g. [XZ], Lemma 4.1$(i)$), and hence $(ii)$ follows from $(i)$. ■

Corollary 4.1$(ii)$ was obtained for $n = 2$ in Theorem 1.1$(i)$ of [LZ2] (see also Proposition 3.6$(i)$ of [IKM]).

4.2. Dually, denote $D(A_A)^\perp$ by $\mathbf{CoCM}(A)$. An $A$-module $G$ is *Gorenstein-injective* ([EJ1]), if there is an exact sequence $\cdots \to I^{-1} \to I^0 \xrightarrow{d^0} I^1 \to \cdots$ of injective $A$-modules, which stays exact under $\mathrm{Hom}_A(D(A_A), -)$, and such that $G \cong \mathrm{Ker}\, d^0$. Let $A$-$\mathcal{G}inj$ be the full subcategory of $A$-mod of Gorenstein-injective modules. Then $A$-$\mathcal{G}inj \subseteq \mathbf{CoCM}(A)$; and if $A$ is Gorenstein then $A$-$\mathcal{G}inj = \mathbf{CoCM}(A)$. By Theorem 3.1'$(v)$ and Corollary 4.1 we have

**Corollary 4.2.** $(i)$ *Let $A$ be an Artin algebra with* $\mathrm{inj.dim}_A A < \infty$. *Then* $\mathbf{CoCM}(T_n(A)) = \mathcal{F}_n(\mathbf{CoCM}(A))$.

$(ii)$ *Let $A$ be a Gorenstein algebra. Then $T_n(A)$-$\mathcal{G}inj = \mathcal{F}_n(A$-$\mathcal{G}inj)$, and the set of $T_n(A)$-modules which are simultaneously Gorenstein-projective and Gorenstein-injective is*

$$\{\mathbf{m}_n(M) \mid M \text{ is simultaneously a Gorenstein-projective and Gorenstein-injective } A\text{-module}\}.$$

4.3. Let $D^b(A)$ be the bounded derived category of $A$, and $K^b(\mathcal{P}(A))$ the bounded homotopy category of $\mathcal{P}(A)$. The singularity category $D^b_{sg}(A)$ of $A$ is defined to be the Verdier quotient $D^b(A)/K^b(\mathcal{P}(A))$. If $A$ is Gorenstein, then there is a triangle-equivalence $D^b_{sg}(A) \cong \underline{\mathbf{CM}(A)}$, where $\underline{\mathbf{CM}(A)}$ is the stable category of $\mathbf{CM}(A)$ modulo $\mathcal{P}(A)$ ([H2], Theorem 4.6; see also [Buc], Theorem 4.4.1). Thus by Corollary 4.1 we have

**Corollary 4.3.** *Let $A$ be a Gorenstein algebra. Then there is a triangle-equivalence*

$$D^b_{sg}(T_n(A)) \cong \underline{\mathcal{S}_n(\mathbf{CM}(A))}.$$

*In particular, if $A$ is a self-injective algebra, then $D^b_{sg}(T_n(A)) \cong \underline{\mathcal{S}_n(A)}$.*

4.4. We have the following characterization of self-injective algebras.

**Theorem 4.4.** *Let $A$ be an Artin algebra. Then $A$ is a self-injective algebra if and only if $T_n(A)$-$\mathcal{G}proj = \mathcal{S}_n(A)$.*

**Proof.** The "only if" part follows from Corollary 4.1$(ii)$. Conversely, by assumption $\begin{pmatrix} D(A_A) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathcal{S}_n(A)$ is a Gorenstein-projective $T_n(A)$-module. Then there is an exact sequence $\cdots \to P^{-1} \to P^0 \xrightarrow{d^0} P^1 \to \cdots$ of projective $T_n(A)$-modules with $\begin{pmatrix} D(A_A) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cong \mathrm{Ker}\, d^0$. By taking the 1-st branch we get an exact sequence $\cdots \to P_1^{-1} \to P_1^0 \xrightarrow{d_1^0} P_1^1 \to \cdots$ of projective $A$-modules with $\mathrm{Ker}\, d_1^0 \cong D(A_A)$. This implies that $D(A_A)$ is a projective module, i.e., $A$ is self-injective. ■

## 5. **Finiteness of monomorphism categories**

This section is to characterize $\mathcal{S}_n(A)$ which is of finite type.

5.1. An additive full subcategory $\mathcal{X}$ of $A$-mod, which is closed under direct summands, is of *finite type* if there are only finitely many isomorphism classes of indecomposable $A$-modules in $\mathcal{X}$. If $A$-$\mathcal{G}proj$ is of finite type, then $A$ is said to be *CM-finite*.

An $A$-module $M$ is *an $A$-generator* if each projective $A$-module is in $\operatorname{add} M$. A $T_n(A)$-generator $M$ is *a bi-generator of* $\mathcal{S}_n(A)$ if $M \in \mathcal{S}_n(A)$ and $\mathbf{m}(D(A_A)) \in \operatorname{add}(M)$.

**Theorem 5.1.** *Let $A$ be an Artin algebra. Then $\mathcal{S}_n(A)$ is of finite type if and only if there is a bi-generator $M$ of $\mathcal{S}_n(A)$ such that* $\operatorname{gl.dim}\operatorname{End}_{T_n(A)}(M) \le 2$.

If $A$ is self-injective, then $\mathbf{m}(D(A_A))$ is a projective $T_n(A)$-module, and hence in $\operatorname{add}(M)$ for each $T_n(A)$-generator $M$. Combining Theorems 5.1 and 4.4 we have

**Corollary 5.2.** *Let $A$ be a self-injective algebra. Then $T_n(A)$ is CM-finite if and only if there is a $T_n(A)$-generator $M$ which is Gorenstein-projective, such that* $\operatorname{gl.dim}\operatorname{End}_{T_n(A)}(M) \le 2$.

Corollary 5.2 also simplifies the result in [LZ1] in this special case.

5.2. The proof of Theorem 5.1 will use Corollary 3.2, and Auslander's idea in proving his classical result cited in Introduction. Given modules $M, X \in A$-mod, denote by $\Omega_M(X)$ the kernel of a minimal right approximation $M' \to X$ of $X$ in $\operatorname{add}(M)$. Define $\Omega_M^0(X) = X$, and $\Omega_M^i(X) = \Omega_M(\Omega_M^{i-1}(X))$ for $i \ge 1$. Define $\operatorname{rel.dim}_M X$ to be the minimal non-negative integer $d$ such that $\Omega_M^d(X) \in \operatorname{add}(M)$, or $\infty$ if otherwise. The following fact is well-known.

**Lemma 5.3.** *(M. Auslander) Let $A$ be an Artin algebra and $M$ be an $A$-module, and $\Gamma = (\operatorname{End}_A(M))^{op}$. Then $\operatorname{proj.dim}{}_\Gamma \operatorname{Hom}_A(M, X) \le \operatorname{rel.dim}_M X$ for all $A$-modules $X$. If $M$ is a generator, then equality holds.*

For an $A$-module $T$, denote by $\mathcal{X}_T$ the full subcategory of $A$-mod given by

$$\{X \mid \exists \text{ an exact sequence } 0 \to X \to T_0 \xrightarrow{d_0} T_1 \xrightarrow{d_1} \cdots, \text{with } T_i \in \operatorname{add}(T),\ \operatorname{Ker} d_i \in {}^\perp T,\ \forall\, i \ge 0\}.$$

Note that $\mathcal{X}_T \subseteq {}^\perp T$, and $\mathcal{X}_T = {}^\perp T$ if $T$ is a cotilting module ([AR], Theorem 5.4($b$)).

**Lemma 5.4.** *Let $A$ be an Artin algebra and $M$ be an $A$-generator with $\Gamma = (\operatorname{End}_A(M))^{op}$, and $T \in \operatorname{add}(M)$. Then for every $A$-module $X \in \mathcal{X}_T$ and $X \notin \operatorname{add}(T)$, there is a $\Gamma$-module $Y$ such that* $\operatorname{proj.dim}_\Gamma Y = 2 + \operatorname{proj.dim}_\Gamma \operatorname{Hom}_A(M, X)$.

**Proof.** By $X \in \mathcal{X}_T$ there is an exact sequence $0 \to X \xrightarrow{u} T_0 \xrightarrow{v} T_1$ with $T_0,\ T_1 \in \operatorname{add}(T) \subseteq \operatorname{add}(M)$. This yields an exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(M, X) \xrightarrow{u_*} \operatorname{Hom}_A(M, T_0) \xrightarrow{v_*} \operatorname{Hom}_A(M, T_1) \longrightarrow \operatorname{Coker} v_* \longrightarrow 0.$$

Since the image of $v_*$ is not projective (otherwise, $u_*$ splits, and then one can deduce a contradiction $X \in \mathrm{add}(T)$), putting $Y = \mathrm{Coker}\, v_*$ we have $\mathrm{proj.dim}_\Gamma Y = 2 + \mathrm{proj.dim}_\Gamma \mathrm{Hom}_A(M, X)$. ■

5.3. **Proof of Theorem 5.1.** Assume that $\mathcal{S}_n(A)$ is of finite type. Then there is a $T_n(A)$-module $M$ such that $\mathcal{S}_n(A) = \mathrm{add}(M)$. Since $\mathbf{m}(D(A_A)) \in \mathcal{S}_n(A) = \mathrm{add}(M)$ and $\mathcal{S}_n(A)$ contains all the projective $T_n(A)$-modules, by definition $M$ is a bi-generator of $\mathcal{S}_n(A)$. Put $\Gamma = (\mathrm{End}_{T_n(A)}(M))^{op}$. For every $\Gamma$-module $Y$, take a projective presentation $\mathrm{Hom}_{T_n(A)}(M, M_1) \xrightarrow{f_*} \mathrm{Hom}_{T_n(A)}(M, M_0) \to Y \to 0$ of $Y$, where $M_1, M_0 \in \mathrm{add}(M)$, and $f : M_1 \to M_0$ is a $T_n(A)$-map. Since $\mathrm{inj.dim}\, \mathbf{m}(D(A_A)) = 1$ (Lemma 3.7) and $M_1 \in \mathcal{S}_n(A) = {}^\perp\mathbf{m}(D(A_A))$ (Corollary 3.2), it follows that $\mathrm{Ker}\, f \in {}^\perp(\mathbf{m}(D(A_A))) = \mathrm{add}(M)$. Thus

$$0 \longrightarrow \mathrm{Hom}_{T_n(A)}(M, \mathrm{Ker}\, f) \longrightarrow \mathrm{Hom}_{T_n(A)}(M, M_1) \longrightarrow \mathrm{Hom}_{T_n(A)}(M, M_0) \longrightarrow Y \longrightarrow 0$$

is a projective resolution of $\Gamma$-module $Y$, i.e., $\mathrm{proj.dim}_\Gamma Y \le 2$. This proves $\mathrm{gl.dim}\,\Gamma \le 2$. Since $A$ is an Artin algebra, we have $\mathrm{gl.dim}\,\mathrm{End}_{T_n(A)}(M) = \mathrm{gl.dim}\,\Gamma \le 2$.

Conversely, assume that there is a bi-generator $M$ of $\mathcal{S}_n(A)$ such that $\mathrm{gl.dim}\,\mathrm{End}_{T_n(A)}(M) \le 2$. Put $\Gamma = (\mathrm{End}_{T_n(A)}(M))^{op}$. Then $\mathrm{gl.dim}\,\Gamma \le 2$. We claim that $\mathrm{add}(M) = {}^\perp\mathbf{m}(D(A_A))$, and hence by Corollary 3.2 $\mathcal{S}_n(A)$ is of finite type. In fact, since $M \in \mathcal{S}_n(A) = {}^\perp\mathbf{m}(D(A_A))$, it follows that $\mathrm{add}(M) \subseteq {}^\perp\mathbf{m}(D(A_A))$. On the other hand, let $X \in {}^\perp\mathbf{m}(D(A_A))$. By Corollary 3.2 $\mathbf{m}(D(A_A))$ is a cotilting $T_n(A)$-module, and hence ${}^\perp\mathbf{m}(D(A_A)) = \mathcal{X}_{\mathbf{m}(D(A_A))}$, by Theorem 5.4($b$) in [AR]. If $X \in \mathrm{add}(\mathbf{m}(D(A_A)))$, then $X \in \mathrm{add}(M)$ since by assumption $\mathbf{m}(D(A_A)) \in \mathrm{add}(M)$. If $X \notin \mathrm{add}(\mathbf{m}(D(A_A)))$, then by Lemma 5.4 there is a $\Gamma$-module $Y$ such that $\mathrm{proj.dim}_\Gamma Y = 2 + \mathrm{proj.dim}_\Gamma \mathrm{Hom}_{T_n(A)}(M, X)$. Now by Lemma 5.3 we have

$$\mathrm{rel.dim}_M X = \mathrm{proj.dim}_\Gamma \mathrm{Hom}_{T_n(A)}(M, X) = \mathrm{proj.dim}_\Gamma Y - 2 \le \mathrm{gl.dim}\,\Gamma - 2 \le 0,$$

this means $X \in \mathrm{add}(M)$. This proves the claim and completes the proof. ■

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